Math 4200
Final exam review sheet
December 4, 2020

The final exam is this Monday December 7. You may choose any 3-hour time window starting on the hour, between 10:00 a.m. and 4:00 p.m. Send me an email which indicates your preference, at korevaar@math.utah.edu. The format of the exam will be similar to our midterms: There will be a mandatory section of basic concept problems, for 40% of the exam total, followed by a choice of 4 problems from a list of 6 or 7 possibilities. If you choose to the the optional section 5.1-5.2 homework you may use that for 2 of your 4 problem choices. Several questions may ask you to prove key theorems from complex analysis. 30 pts

I will provide the residues table from page 250 in section 4.1, and the contour integration page Table 4.3.1 page 296. But otherwise the exam is closed note/book/internet/friend etc.

I will post the 2011 practice final exam and solutions - which did not have the same format, however. Make sure that in addition to the homework, you review our first two midterms and the quizzes, to get ideas for many of the sorts of questions which may appear.

I think of basic complex analysis as following from (only) several key ideas, and the core of these ideas is what I hope you carry away from the course, along with the knowledge that interesting complex analysis magic appears in a lot of different areas of higher math. Here are the core ideas:

(0) Complex number algebra and geometry: addition, multiplication, conjugation, inverse, Euler's formula and the polar form of complex numbers. It is essential to understand this algebra and geometry, in order to understand the basic analytic functions.

(1) $f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$ is complex differentiable at z_0 (the limit definition of complex derivative) iff $F(x, y) = (u(x, y), v(x, y))$ is real differentiable, with derivative matrix a rotation dilation. This circle of ideas includes the Cauchy Riemann equations, the chain rule for $f(\gamma(t))$, where $f(z)$ is analytic and $\gamma(t)$ is a parametric curve. It also includes the calculus of complex differentiation and the inverse function theorem for analytic functions, and leads to the geometry of conformal transformations.

(2) Conformal transformations (bijections) between specified open, connected domains. This includes the zoo of transformations we studied in chapter 1, including the concepts of branch points and branch cuts. We returned to these ideas at the very end of the course, with the fractional linear transformations and Riemann Mapping Theorem in sections 5.1-5.2.

(3) Contour integrals of functions,

$$\int_{\gamma} f(z)\, dz,$$

how to compute by parameterization; the real and imaginary part are each real line integrals; why such integrals equal zero when f is analytic and γ is the oriented boundary of a domain (Green's Theorem and CR equations!... Green's Theorem also yields alternate proofs of Cauchy's Theorem, the Cauchy integral formula and the residue theorem in "simple" cases.)

(4) The circle of theorems related to Cauchy's Theorem and the Cauchy integral formula, inclulding their statements and proofs:
(4a) f analytic in a disk ==> rectangle theorem in disk ==> antiderivative in disk.
local antiderivatives implies the deformation theorem for two homotopic closed curves in an domain A (which says that the contour integrals over the two curves must agree). In particular, for a simply connected domain, contour integrals of analytic functions over closed curves must be zero, implying global antiderivatives exist.
(4b) definition of index, computation via a contour integral, and the derivation of the Cauchy integral formula, using (4a).
(4c) CIF for derivatives. Liouville. FTA. mean value and maximum modulus principle properties for analytic functions.
(4d) Mean value property for harmonic functions, maximum and minimum principles, harmonic conjugates in simply connected domains.

(5) Power series and Laurent series.
(5a) radius of convergence for power series (and uniform absolute convergence inside), and complementary result for power series in negative powers. Annulus of convergence consequence for a series with positive and negative powers. The fact that the resulting sums are analytic, using the Weierstrass M test and the fact that uniform limits of analytic functions are analytic, with limit derivative equal the limit of the derivatives. (uses Morera, CIF!).
(5b) derivation and uniqueness of power series and Laurent series for analytic functions in disks and annuli. (The amazing uses of geometric series.)
(5c) uniqueness of analytic extension ... based on the theorem that zeroes of analytic functions are isolated, so if two analytic functions agree on a sequence converging to a point not in the sequence, then they agree on their common domain.
(5d) radius of convergence for power and Laurent series, based on explicit convergence tests or on domain of analyticity of a function.
(5e) finding power and Laurent series for given functions, e.g. method of equating coefficients in products or quotients...and you should know the key power series (trig, geometric, power etc.) if you hope to get started!

(6) Residue calculus
(6a) computing residues. (I will provide you with the text's table of residue computation tools, but we've discussed the key ideas which often provide shortcuts and more effective ways to compute these.)
(6b) statement and proof of the residue theorem.
(6c) computing integrals with the residue theorem, or with some limiting process which starts with the residue theorem. (I will provide you with the table on page 296 of the text, but you will need to justify the usage of any such formula with estimates, when appropriate.)
(6d) summing series with residue calculus. (See homework and class notes for representative examples....We just touched on magic formulas for "infinite partial fractions", and as we've seen in the Riemann Zeta and thetat function presentations, there are related magic product formulas with which one can often express functions with inifinitely many zeroes as infinite products and sums.)